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Periodicity of P-adic Continued Fraction Expansions

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In 1940, K. Mahler presented a geometric algorithm which, for any P-adic integer ζ, yields a sequence of pairs of integers (p_n, q_n) which give P-adic approximations that are best with respect to $\Phi(X, Y)$, a real, reduced, positive-definite quadratic form of determinant -1. The algorithm also constructs a sequence of 2×2 integer matrices of determinant P, denoted $\Omega(\zeta)$, which defines the pairs (p_n, q_n) from a product of the first n matrices of $\Omega(\zeta)$. In this paper, an equivalence relation is considered which relates P-adic integers ζ and ξ if their sequences $\Omega(\zeta)$ and $\Omega(\xi)$ eventually agree. This is shown to happen if and only if $\zeta = T\xi$, where T is an integer transformation of determinant P^g which satisfies specified conditions. Mahler showed that if $\Omega(\zeta)$ is periodic then ζ is rational or a quadratic irrational, yet for such a quadratic irrational, Φ can be chosen so that $\Omega(\zeta)$ is no longer periodic. In this paper, the quadratic irrationals ζ, for which $\Omega(\zeta)$ is periodic for some choice of Φ, are characterized. A relation similar to equivalence is used in the proof. In particular, it is concluded that there are quadratic irrationals ζ for which $\Omega(\zeta)$ is never periodic for any Φ.

1. INTRODUCTION

In 1940, Mahler [4] presented a geometric algorithm which yields sequences of pairs of integers which give best P-adic approximations to P-adic integers, i.e., for $\Phi(X, Y)$ a real reduced positive definite quadratic form of determinant -1, and for a P-adic integer ζ, the algorithm defines pairs (p_n, q_n) such that $|q_n\zeta + p_n|_P \leqslant P^{-n}$, and that if a pair (p, q) satisfies $0 < \Phi(p, q) < \Phi(p_n, q_n)$ then $|q\zeta + p|_P > |q_n\zeta + P_n|_P$. The algorithm also construct a sequence of 2×2 integer matrices of determinant P, denoted $\Omega(\zeta)$, which defines the pairs (p_n, q_n) from the product of the first n matrices in $\Omega(\zeta)$.

This corresponds to the theory of real continued fractions where the partial quotients can be used to construct a sequence of 2×2 integer matrices

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of determinant -1 from which the nth best rational approximation is calculated as entries of a product of the first n matrices. Real numbers are said to be equivalent when their continued fraction expansions eventually agree; this happens if and only if the real numbers are related by an linear fractional transformation with integer coefficients of determinant ± 1 (Serret). This equivalence is used in showing that a real continued fraction expansion is periodic if and only if the number is a quadratic irrational (Lagrange). Equivalence is also useful for studying best Diophantine approximation constants.

In this paper, an equivalence relation is considered which relates P-adic integers ζ and ξ if their sequences $\Omega(\zeta)$ and $\Omega(\xi)$ eventually agree. This is shown to happen if and only if $\zeta = T\xi$, where T is an integer transformation of determinant P^g which satisfies specified conditions (Theorem 5).

Mahler showed that if $\Omega(\zeta)$ is periodic then ζ is rational [4, Sect. 17] or ζ is a quadratic irrational; yet if ζ is a quadratic irrational for which $\Omega(\zeta)$ is periodic, it is possible to choose Φ so that $\Omega(\zeta)$ is no longer periodic [4, Sect. 10]. In this paper, the quadratic irrationals ζ, for which $\Omega(\zeta)$ is periodic for some choice of Φ, are characterized. A relation similar to equivalence is used in the proof. In particular, it is concluded that there are quadratic irrationals ζ for which $\Omega(\zeta)$ is never periodic for any choice of Φ.

Recently, Mahler's algorithm has been generalized by de Weger [5] to other norms. Periodicity in that context is reported in his paper.

2. Mahler's Algorithm for P-adic Diophantine Approximations

Mahler [4] presented an algorithm which yields sequences of approximations to P-adic integers by rational numbers which are best with respect to a real reduced positive definite quadratic form of determinant -1. In this section, a brief geometric description of the algorithm is given. For details, see [4, 1].

Let λ be a fixed complex number in F, the fundamental domain of the modular group, i.e., the set of complex numbers in the upper half-plane which satisfies

$$-\tfrac{1}{2} \leqslant \operatorname{Re} z < \tfrac{1}{2} \quad \text{and} \quad |z| > 1 \qquad \text{or} \qquad -\tfrac{1}{2} \leqslant \operatorname{Re} z \leqslant 0 \quad \text{and} \quad |z| = 1.$$

Set $\Phi(X, Y) = (1/\operatorname{Im} \lambda)(X - \lambda Y)(X - \bar{\lambda} Y) = (1/\operatorname{Im} \lambda)|X - \lambda Y|^2$. Let ζ be a P-adic integer. For every $n > 0$, let A_n be the unique (rational) integer which satisfies $0 \leqslant A_n < P^n$ and $\zeta \equiv A_n \pmod{P^n}$.

The algorithm defines three sequences. The first of these, $z(\zeta)$, is a sequence of complex numbers. Let $z_n \in z(\zeta)$ be defined as the unique complex number in F which is equivalent by an element of the modular group to $(A_n+\lambda)/P^n$. That is, for some integer matrix $\left[\begin{smallmatrix} r_n & r'_n \\ q_n & q'_n \end{smallmatrix}\right]$ of determinant 1,

$$\frac{A_n+\lambda}{P^n}=\frac{r_n z_n + r'_n}{q_n z_n + q'_n}.$$

The second sequence $T(\zeta)$ is a sequence of 2×2 integer matrices where, for each $n\geqslant 0$, $T_n\in T(\zeta)$ is defined by

$$T_n=\begin{bmatrix} p_n & p'_n \\ q_n & q'_n \end{bmatrix}, \qquad \begin{aligned} p_n &= r_n P^n - q_n A_n, \\ p'_n &= r'_n P^n - q'_n A_n. \end{aligned}$$

Then, T_n has determinant P^n and the action of T_n as a linear fractional transformation is

$$T_n z_n=\lambda \qquad \text{for all } n\geqslant 0. \tag{1}$$

Note that by the construction,

$$(q_n, q'_n)=1. \tag{2}$$

It is clear that $|q_n\zeta+p_n|_P\leqslant P^{-n}$. Mahler showed in [4] that if $|q\zeta+p|_P\leqslant P^{-n}$, $\Phi(p,q)>0$, then $\Phi(p,q)\geqslant\Phi(p_n,q_n)$. Thus $T(\zeta)$ determines approximations to ζ which are best with respect to $\Phi(X,Y)$.

To study the relationship between the successive elements of $z(\zeta)$, a third sequence is introduced. Set

$$\Omega_n=T_{n-1}^{-1}T_n \qquad \text{all } n\geqslant 1.$$

Each Ω_n is a matrix of determinant P with integers entries. The sequence $T(\zeta)$ can be recovered from the sequence $\Omega(\zeta)$ because

$$T_n=\Omega_1\Omega_2\cdots\Omega_n, \qquad n\geqslant 1.$$

If $\Omega_n\in\Omega(\zeta)$ is considered as a transformation, then

$$\Omega_n z_n=z_{n-1}, \qquad n\geqslant 1.$$

Let $M(P)$ denote the set of 2×2 integer matrices of determinant P which satisfy $\Omega F\cap F\neq\varnothing$, where Ω acts as a transformation. Any matrix which can occur in the sequence $\Omega(\zeta)$ for some P-adic integer ζ and some $\lambda\in F$ is necessarily in $M(P)$. Mahler showed that the set $M(P)$ is finite for any P and that any $\Omega\in M(P)$ can occur as some $\Omega_n\in\Omega(\zeta)$ for some P-adic integer ζ (although possibly only for special choices of λ).

The following result of Mahler gives necessary and sufficient criteria for determining when a sequence of matrices of determinant P is equal to the sequence $\Omega(\zeta)$ for some P-adic integer ζ.

THEOREM 1 (Mahler). *Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of 2×2 integer matrices of determinant P. Define a second sequence $\{T_n\}_{n=0}^{\infty}$ of matrices where*

$$T_0=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}p_0 & p_0'\\ q_0 & q_0'\end{bmatrix},$$

$$T_n=\Omega_1\Omega_2\cdots\Omega_n=\begin{bmatrix}p_n & p_n'\\ q_n & q_n'\end{bmatrix}\qquad \text{for all } n\geqslant 1.$$

Then there exists a P-adic integer ζ such that $\{\Omega_n\}=\Omega(\zeta)$ and $\{T_n\}=T(\zeta)$ if and only if $(q_n'\lambda-p_n')/(-q_n\lambda+p_n)\in F$ and $(q_n, q_n')=1$ for all $n\geqslant 0$.

The conditions are necessary as noted in Eqs. (1) and (2).

COROLLARY (Mahler). *Let $T=[\begin{smallmatrix}p & p'\\ q & q'\end{smallmatrix}]$ be an integer matrix of determinant P^n such that $(q'\lambda-p)/(-q\lambda+p)\in F$* and $(q,q')=1$. *Then there exists a P-adic integer ζ such that $T=T_n\in T(\zeta)$.*

3. UNIQUENESS IN MAHLER'S ALGORITHM

It has already been noted that for a fixed λ, the sequence $z(\zeta)$ is uniquely determined for any P-adic integer ζ. It should be pointed out that this is not always true of the sequences $T(\zeta)$ and $\Omega(\zeta)$ or of the sequence of approximations (p_n, q_n). Mahler discussed uniqueness in the sequence of approximations in his Theorem 18 of [4]. Proposition 2 is a slightly stronger result needed for this paper. The uniqueness of a transformation (or an approximating pair) is up to multiplication of all entries by -1.

PROPOSITION 2. *For any P-adic integer ζ, the transformation $T_n\in T(\zeta)$ is uniquely determined for an index n if and only if z_n is different from $\sqrt{-1}$ and $(-1+\sqrt{-3})/2$; the transformation $\Omega_n\in\Omega(\zeta)$ is uniquely determined for an index n if and only if z_{n-1} and z_n in $z(\zeta)$ are both different from $\sqrt{-1}$ and $(-1+\sqrt{-3})/2$. The approximating pair (p_n, q_n) is uniquely determined for an index n if and only if z_n in $z(\zeta)$ is different from $\sqrt{-1}$ and $(-1+\sqrt{-3})/2$. If $z_n=\sqrt{-1}$ then $\pm(p_n', q_n')$ may occur as the approximating pair; if $z_n=(-1+\sqrt{-3})/2$ then $\pm(p_n', q_n')$ or $\pm(-p_n+p_n', -q_n+q_n')$ may occur as the approximating pair.*

Proof. The point $z_n\in z(\zeta)$ is the unique point in F which is equivalent by a unimodular transformation S with integer coefficients to $(A_n+\lambda)/P^n$, say $Sz_n = (A_n + \lambda)/P^n$. Since $T_n = \begin{bmatrix} P^n & -A_n \\ 0 & 1 \end{bmatrix} S$, the uniqueness of T_n depends on the uniqueness of S. Because every point in the upper half-plane is equivalent to exactly one point in F, the transformation is not uniquely determined if and only if z_n is equivalent to itself by some nonidentity unimodular transformation.

It is well known (see [3, p. 19]) that the only points $z \in F$ and unimodular transformations S which satisfy $Sz = z$ are $z = \sqrt{-1}$ and $S = \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ or $z = (-1 + \sqrt{-3})/2$ and $S = \pm \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\pm 1}$.

Thus if $z_n \neq \sqrt{-1}$, $(-1 + \sqrt{-3})/2$, then the transformation which maps z_n to $(A_n + \lambda)/P^n$ is uniquely determined (up to ± 1), and hence T_n is uniquely determined.

Consequently, if $z_n = \sqrt{-1}$ with $T_n z_n = \lambda$, then also $\pm T_n \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} z_n = \lambda$. If $z_n = (-1 + \sqrt{-3})/2$ with $T_n z_n = \lambda$, then also $\pm T_n \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\pm 1} z_n = \lambda$.

The transformation $\Omega_n = T_{n-1}^{-1} T_n$, thus Ω_n is uniquely determined if and only if both T_{n-1} and T_n are. This happens if and only if z_{n-1} and z_n are both different from $\sqrt{-1}$ and $(-1 + \sqrt{-3})/2$.

The approximating pair (p_n, q_n) is uniquely determined if and only if T_n is. The approximating pairs possible when T_n is not unique for some index n can be read from the possible alternative(s) for T_n.

4. Equivalence of P-adic Integers

Definition. For a fixed λ, two P-adic integers ζ and ξ are *equivalent* when, for some nonnegative integers m and n,

$$\Omega_{m+i}(\zeta) = \Omega_{n+i}(\xi)$$

for all $i > 0$.

Assume that the P-adic integers ζ and ξ are equivalent with m and n as in the definition. Recall that $T_{m+i}(\zeta) = \Omega_1(\zeta)\,\Omega_2(\zeta) \cdots \Omega_{m+i}(\zeta)$ and $T_{n+i}(\xi) = \Omega_1(\xi)\,\Omega_2(\xi) \cdots \Omega_{n+i}(\xi)$. Set

$$T = P^{-k} T_m(\zeta)(P^n T_n^{-1}(\xi)),$$

where k is sufficiently large so that the integer entries of T have no common factor. The determinant of T is P^{n+m-2k}. This T satisfies

$$P^{n-k} T_{m+i}(\zeta) = T T_{n+i}(\xi) \tag{3}$$

for all $i \geqslant 0$. Because the right-hand side of (3) is an integer matrix and the entries of $T_{m+i}(\zeta)$ have no common factor, $k \leqslant n$. Similarly,

$$P^{m-k} T_{n+i}(\xi) = (P^{m+n-2k} T^{-1})\, T_{m+i}(\zeta)$$

yields $k \leqslant m$. Hence $k \leqslant \min\{m, n\}$.

Conversely, if there exists an integer matrix T which satisfies (3) for some nonnegative integers m, n, and k with $k \leqslant \min\{m, n\}$ and all $i \geqslant 0$, then ζ and ξ are equivalent.

5. Necessary and Sufficient Conditions for Equivalence

The following two lemmas lead to a theorem giving necessary and sufficient conditions which must be satisfied by a matrix T relating two equivalent P-adic integers.

LEMMA 3. *Let ζ and ξ be P-adic integers and let $T_n = T_n(\xi) = \left[\begin{smallmatrix} p_n & p'_n \\ q_n & q'_n \end{smallmatrix}\right]$ (for some λ). Let $T = \left[\begin{smallmatrix} p & p' \\ q & q' \end{smallmatrix}\right]$ be a matrix of determinant P^g for some $g \geqslant 0$. Let*

$$TT_n = T_n^* = \begin{bmatrix} p^*{}_n & p^{*\prime}{}_n \\ q^*{}_n & q^{*\prime}{}_n \end{bmatrix}.$$

Then $-p^*{}_n/q^*{}_n$ and $-p^{*\prime}{}_n/q^{*\prime}{}_n$ converge P-adically to

$$\zeta = \frac{-p\xi + p'}{q\xi - q'} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p & p' \\ q & q' \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \xi.$$

Specifically, $|q^*{}_n\zeta + p^*{}_n|_P = (|q_n\xi + p_n|_P / |q\xi - q'|_P)\, P^{-g}$ and $|q^{*\prime}{}_n\zeta + p^{*\prime}{}_n|_P = (|q'_n\xi + p'_n|_P / |q\xi - q'|_P)\, P^{-g}$.

Proof. For all $n \geqslant 0$,

$$\begin{aligned} q^*{}_n\zeta + p^*{}_n &= (p_n q + q_n q')\frac{-p\xi + p'}{q\xi - q'} + (p_n p + q_n p') \\ &= \frac{(p_n q + q_n q')(-p\xi + p') + (p_n p + q_n p')(q\xi - q')}{(q\xi - q')} \\ &= \frac{(q_n\xi + p_n)}{(q\xi - q')}(p'q - pq'). \end{aligned}$$

The computation for $q^{*\prime}{}_n\zeta + p^{*\prime}{}_n$ is similar.

Notation. For any matrix T, the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] T \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ will be denoted $\hat{T}$.

Note that, as transformations, $\widehat{T_1 T_2} = \hat{T}_1 \hat{T}_2$ and $\hat{T}^{-1} = (\widehat{T^{-1}})$.

LEMMA 4. *Let $\left[\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right]$ be an integer matrix and let ζ be a P-adic integer. Then $\left[\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right] = T_n(\zeta) M$, for some integer matrix M, if and only if*

$$|y\zeta + x|_P \leqslant P^{-n} \qquad \text{and} \qquad |y'\zeta + x'|_P \leqslant P^{-n}.$$

Proof. Write $T_n(\zeta)=\left[\begin{smallmatrix} p_n & p'_n \\ q_n & q'_n \end{smallmatrix}\right]$ and $M=\left[\begin{smallmatrix} r & r' \\ s & s' \end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right]=T_n(\zeta)\,M$ yields

$$\begin{aligned} |y\zeta+x|_P &= |r(q_n\zeta+p_n)+s(q'_n\zeta+p'_n)|_P \\ &\leqslant \max\{|q_n\zeta+p_n|_P, |q'_n\zeta+p'_n|_P\}\leqslant P^{-n}, \end{aligned}$$

and likewise for $|y'\zeta+x'|_P$. Conversely, when $|y\zeta+x|_P\leqslant P^{-n}$ and $|y'\zeta+x'|_P\leqslant P^{-n}$, it is necessary to verify that

$$M=(T_n(\zeta))^{-1}\begin{bmatrix} x & x' \\ y & y' \end{bmatrix}=\frac{1}{P^n}\begin{bmatrix} q'_n x-p'_n y & q'_n x'-p'_n y' \\ -q_n x+p_n y & -q_n x'+p_n y' \end{bmatrix}$$

has integer entries. For the upper left entry, the integer $q'_n x-p'_n y$ satisfies

$$\begin{aligned} |q'_n x-p'_n y|_P &= |q'_n(x+y\zeta)-y(q'_n\zeta+p'_n)|_P \\ &\leqslant \max\{|x+y\zeta|_P, |q'_n\zeta+p'_n|_P\}\leqslant P^{-n} \end{aligned}$$

and is thus divisible by P^n. Hence $(q'_n x-p'_n y)/P^n\in Z$. Similar computations show that all entries of $(T_n(\zeta))^{-1}\left[\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right]$ are integers.

THEOREM 5. *Let ζ and ξ be P-adic integers. Let m, n and k be nonnegative integers with $k\leqslant\min\{m,n\}$. Let T be a matrix of determinant P^{n+m-2k} whose entries are not all divisible by P. Then ζ and ξ are equivalent if and only if, for $T=\left[\begin{smallmatrix} p & p' \\ q & q' \end{smallmatrix}\right]$,*

(i) $\zeta=\hat{T}\xi$,

(ii) $T^{-1}\lambda\in\bigcap_{i=0}^{\infty}T_{n+i}(\xi)\,F$,

(iii) $\min\{\operatorname{ord}_P(q\xi-q'), \operatorname{ord}_P(p\xi-p')\}=\operatorname{ord}_P(q\xi-q')=n-k.$

Remarks. This theorem discusses characteristics of a matris T which relates two equivalent P-adic integers; thus the conditions (i)–(iii) imply their symmetric forms

(i′) $\xi=\hat{T}^{-1}\zeta=(\widehat{P^{n+m-2k}T^{-1}})\,\zeta$,

(ii′) $T\lambda\in\bigcap_{i=0}^{\infty}T_{m+i}(\zeta)\,F$,

(iii′) $\min\{\operatorname{ord}_P(q\zeta+p), \operatorname{ord}_P(q'\zeta+p')\}=\operatorname{ord}_P(q\zeta+p)=m-k.$

Condition (ii) is not an impossible criterion. For example, if the matrix T has λ as a fixed point, then the condition is satisfied. Indeed the set $\bigcap_{i=0}^{\infty}T_{n+i}(\xi)\,F$ may have a nonempty interior giving a large selection of possible T's and λ's which work. This will be disucssed in more detail in the next section.

Proof of Theorem. Assume that ζ and ξ are equivalent so that, by the results at the beginning of this section,

$$P^{n-k}T_{m+i}(\zeta)=TT_{n+i}(\xi) \qquad \text{for all } i\geqslant 0. \tag{4}$$

Condition (i) is implied because by Lemma 3, $\zeta=\hat{T}\xi$. For (ii), note that since for all $j\geqslant 0$,

$$\lambda\in T_j(\zeta)\,F=P^{n-k}T_j(\zeta)\,F$$

and from (4)

$$P^{n-k}T_{m+i}(\zeta)\,F=TT_{n+i}(\xi)\,F,$$

it follows that $T^{-1}\lambda\in T_{n+i}(\xi)\,F$ for all $i\geqslant 0$.

For (iii),[1] it follows from (i) that

$$\zeta=\frac{-p\xi+p'}{q\xi-q'}. \tag{5}$$

ζ is a P-adic integer, so $0\leqslant \operatorname{ord}_P(\zeta)=\operatorname{ord}_P(-p\xi+p'/q\xi-q')=\operatorname{ord}_P(-p\xi+p')-\operatorname{ord}_P(q\xi-q')$. Thus $\min\{\operatorname{ord}_P(p\xi-p'), \operatorname{ord}_P(q\xi-q')\}=\operatorname{ord}_P(q\xi-q')$.

It remains to show that $\operatorname{ord}_P(q\xi-q')=n-k$. For $i\geqslant 0$ let

$$T_{n+i}(\xi)=\begin{bmatrix} p_{n+i} & p'_{n+i} \\ q_{n+i} & q'_{n+i}\end{bmatrix} \qquad \text{and} \qquad T_{m+i}(\zeta)=\begin{bmatrix} p^{\#}_{n+i} & p'^{\#}_{n+i} \\ q^{\#}_{n+i} & q'^{\#}_{n+1}\end{bmatrix}.$$

Then

$$|p_{n+i}+q_{n+i}\xi|_P\leqslant P^{-(n+i)}, \tag{6a}$$

$$|p'_{n+i}+q'_{n+i}\xi|_P\leqslant P^{-(n+i)}, \tag{6b}$$

and equality holds for at least one of the above. Similarly,

$$|p^{\#}_{m+i}+q^{\#}_{m+i}\zeta|_P\leqslant P^{-(m+i)}, \tag{7a}$$

$$|p'^{\#}_{m+i} \quad q'^{\#}_{m+i}\zeta|_P\leqslant P^{-(m+i)}, \tag{7b}$$

and equality holds for at least one of the above. By (4) and (5)

$$P^{n-k}(p^{\#}_{m+i}+q^{\#}_{m+i}\zeta)=\frac{-\det T}{q\xi-q'}(p_{n+i}+q_{n+i}\xi),$$

$$P^{n-k}(p'^{\#}_{m+i}+q'^{\#}_{m+i}\zeta)=\frac{-\det T}{q\xi+q'}(p'_{n+i}+q'_{n+i}\xi).$$

[1] This proof that equivalence implies condition (iii) was suggested by the referee.

Hence

$$(n-k)-\operatorname{ord}_P\left(\frac{-\det T}{q\xi-q'}\right)=\operatorname{ord}_P(p_{n+i}+q_{n+i}\xi)-\operatorname{ord}_P(p^{\#}_{m+i}+q^{\#}_{m+i}\zeta) \quad (8)$$
$$=\operatorname{ord}_P(p'_{n+i}+q'_{n+i}\xi)-\operatorname{ord}_P(p'^{\#}_{m+i}+q'^{\#}_{m+i}\zeta).$$

From this it is clear that equality must hold simultaneously in (6a) and (7a) or in (6b) and (7b). The differences in (8) are therefore all equal to $(n+i)-(m+i)=n-m$. It now follows from $(n-k)-\operatorname{ord}_P(-\det T/(q\xi-q')=n-m$ that $\operatorname{ord}_P(q\xi-q')=(n-m)+(n+m-2k)-(n-k)=n-k$. This completes the portion of the proof which shows that T must necessarily satisfy conditions (i)–(iii) if T relates two P-adic integers as in (4).

Now assume that conditions (i)–(iii) are satisfied. In (iii), $\min\{\operatorname{ord}_P(q\xi-q'), \operatorname{ord}_P(p\xi-p')\} = \operatorname{ord}_P(q\xi-q') = n-k$. Therefore $|q\xi-q'|_P=P^{k-n}$ and $|-p\xi+p'|_P\leqslant P^{k-n}$. By Lemma 4,

$$P^{n+m-2k}T^{-1}=\begin{bmatrix} q' & -p' \\ -q & p \end{bmatrix}=T_{n-k}(\xi)\,M$$

for some integer matrix M which must have determinant P^{m-k}. So for all $j\geqslant n-k$,

$$TT_j(\xi)=[(P^{m-k}M^{-1})(P^{n-k}T_{n-k}^{-1}(\xi))][T_{n-k}(\xi)\,\Omega_{n-k+1}(\xi)\cdots\Omega_j(\xi)]$$
$$=(P^{m-k}M^{-1})\,P^{n-k}(\Omega_{n-k+1}(\xi)\cdots\Omega_j(\xi)).$$

Hence for all $j\geqslant n-k$, all entries of $TT_j(\xi)$ are divisible by P^{n-k}. So consider the integer matrices

$$\begin{bmatrix} p^*_j & p'^*_j \\ q^*_j & q'^*_j \end{bmatrix}=T^*_j=\frac{1}{P^{n-k}}\begin{bmatrix} p & p' \\ q & q' \end{bmatrix}\begin{bmatrix} p_j & p'_j \\ q_j & q'_j \end{bmatrix}, \qquad j\geqslant n-k.$$

Using (i), the results of Lemma 3 and (iii),

$$|q^*_j\zeta+p^*_j|_P=P^{n-k}\frac{|p'q-pq'|_P}{|q\xi-q'|_P}|q_j\xi+p_j|_P$$
$$=P^{n-k}\frac{P^{-n-m+2k}}{P^{-n+k}}|q_j\xi+p_j|_P$$
$$\leqslant P^{n-m-j}.$$

Similarly,

$$|q'^*_j\zeta+p'^*_j|_P\leqslant P^{n-m-j}.$$

Furthermore,

$$\det T^*_j = \frac{1}{P^{2(n-k)}} \det T \det T_j(\xi) = P^{m+j-n}.$$

Thus by Lemma 4,

$$T^*_j = T_{m+j-n}(\zeta)\, S_{j-n}$$

for some unimodular integer matrix S_{j-n}, for every $j \geq n-k$.

Now set $j = n+i$. Then

$$TT_{n+i}(\xi) = P^{n-k} T^*_{n+i} = P^{n-k} T_{m+i}(\zeta)\, S_i. \tag{9}$$

By (ii), for all $i \geq 0$, $T^{-1}\lambda \in T_{n+i}(\xi)\, F$ so

$$\lambda \in TT_{n+i}(\xi)\, F = P^{n-k} T_{m+i}(\zeta)\, S_i F = T_{m+i}(\zeta)\, S_i F,$$

or

$$(T_{m+i}(\zeta))^{-1}\lambda \in S_i F.$$

But because $T_{m+i}(\zeta) \in T(\zeta)$, by Theorem 1,

$$(T_{m+i}(\zeta))^{-1}\lambda \in F.$$

Together the above two statements give

$$(T_{m+i}(\zeta))^{-1}\lambda \in S_i F \cap F.$$

Because F is the fundamental domain for the group of unimodular matrices, this intersection is nonempty if $S_i = \pm\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ when $S_i F \cap F = \sqrt{-1}$, or $S_i = \pm\left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]^{\pm 1}$, when $S_i F \cap F = (-1+\sqrt{-3})/2$. By Proposition 2, when $\sqrt{-1} = (T_{m+i}(\zeta))^{-1}\lambda = z_{m+i} \in z(\zeta)$, the matrix $T_{m+i}(\zeta)$ is unique up to multiplication by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$; when $(-1+\sqrt{-3})/2 = (T_{m+i}(\zeta))^{-1}\lambda = z_{m+i} \in z(\zeta)$, the matrix $T_{m+i}(\zeta)$ is unique up to multiplication by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]^{\pm 1}$. Thus for $i \geq 0$, (9) becomes

$$TT_{n+i}(\xi) = P^{n-k} T_{m+i}(\zeta)\, S_i = P^{n-k} T_{m+i}(\zeta).$$

This finishes the proof of the theorem.

6. Periodic Expansions

By Lemma 4, for a P-adic integer ζ and a fixed index n, the $T_n(\zeta)$ for different λ are all related by (or differ by) multiplication by a unimodular integer matrix. In general, no particular choice of λ seems canonical. However, if there is a λ for which the sequence $\Omega(\zeta)$ is periodic, then such a λ is distinctive for that ζ. In this section, the periods which can occur (according to the restrictions of Theorem 1) and the P-adic integers they represent are discussed. Results in Sections 7–10 give a characterization of irrational P-adic integers ζ for which $\Omega(\zeta)$ can be periodic.

DEFINITION. For a P-adic integer ζ, the sequence $\Omega(\zeta)$ is *periodic* if, for some integers $n \geqslant 0$ and $g > 0$, $\Omega_{g+m} = \Omega_m$ for all $m > n$; the sequence is *purely periodic* if it is periodic with $n = 0$. The matrix product $\Omega_{n+1} \cdots \Omega_{n+g} = U$ is a *periodic matrix* of the sequence.

Any real 2×2 matrix U is either upper triangular or, as a transformation, has finite fixed points. If a period matrix U of a sequence $\Omega(\zeta)$ is upper triangular, then ζ is necessarily rational. (See [4, Theorem 11]. This case will not be discussed in this paper.

The case where a period matrix U has finite fixed points requires information about the geometry of tranformations. The basic results quoted and used below are discussed in detail in Chapter 1 of [2].

Suppose a transformation $U = [\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}]$ with real entries has finite fixed points f and f'. If $f = f'$ then this fixed point is real. The action of U on the upper half-plane can be written as

$$\frac{1}{Uz - f} = \frac{1}{z - f} + (\operatorname{sgn}(a + b'))\, b.$$

Such a U is called parabolic. Circles which are fixed under the action of U are all tangent to the real line at f.

If the fixed points are distinct then the action of U on the upper half-plane can be described by

$$\frac{Uz - f}{Uz - f'} = \upsilon \frac{z - f}{z - f'},$$

where

$$\upsilon = \frac{a - bf}{a - bf'} \neq 1$$

is the *multiplier* of the transformation U. There are two cases. If f and f' are real, then υ is real. U is called hyperbolic. The circles which are fixed under the action of U are all circles passing through f and f'; the portion in the upper half-plane of each such circle is mapped into itself. If f and f' are not real then $f' = \bar{f}$, thus $|\upsilon| = 1$. U is called elliptic. The circles which are fixed under an elliptic U form the family of circles which are orthogonal to the circles passing through f and f'; the circle fixed by U which contains a point z_0 is the set of all points z satisfying

$$\left|\frac{z-f}{z-f'}\right| = \left|\frac{z_0-f}{z_0-f'}\right|.$$

THEOREM 6. *Let $U = [\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}]$ be an integer matrix of determinant P^g, $g > 0$. If U is a period matrix of $\Omega(\zeta)$ for an irrational P-adic integer ζ then U must be elliptic with a fixed point in F and must satisfy* $(a+b') \not\equiv 0 \pmod{P}$.

First, the geometry of transformations is used to show that U is elliptic with a fixed point in F and the multiplier of U is not a root of unity. Then a technical lemma from Mahler [4] is used to complete the proof.

Proof. Suppose $\Omega(\zeta)$ is periodic with period $U = \Omega_{n+1} \cdots \Omega_{n+g}$, where n is a fixed index. Since $Uz_{n+kg} = z_{n+(k-1)g}$, it follows that $z_{n+kg} = U^{-k}z_n$. One technique which demonstrates the unsuitability of a particular type of U is to show that, for any $z \in F$, there exists a positive integer k so that $U^{-k}z \notin F$. This technique is used repeatedly.

If U is parabolic, then for any $z \in F$, the circle fixed by U which passes through z is not completely contained in F. The sequence of points z, $U^{-1}z$, $U^{-2}z$,..., advance around this circle toward the fixed point f outside F. Thus for all k sufficiently large, $U^{-k}z \notin F$.

If U is hyperbolic, then again for any $z \in F$, the circle fixed by U which passes through z is not completely contained in F. The sequence of points z, $U^{-1}z$, $U^{-2}z$,..., advance around the circle toward one of the fixed points outside F. Thus for all k sufficiently large, $U^{-k}z \notin F$.

If U is elliptic and its multiplier is a root of unity, a different kind of problem arises. Suppose that, υ, the multiplier of U, is an mth root of unity $(m > 1)$. Then the action of U^m on the upper half-plane is $(U^mz - f)/(U^mz - f') = \upsilon^m(z-f)/(z-f') = (z-f)/(z-f')$, the action of the identity transformation. Thus $U^m = P^{mg}[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$ which would force $T_{n+mg}(\zeta) = T_n(\zeta)\, U^m = [\begin{smallmatrix} p & p' \\ q & q' \end{smallmatrix}]$ to have $(q, q') \geqslant P^{mg} > 1$. This violates the conditions of Theorem 1. Hence if U is elliptic, its multiplier must not be a root of unity.

Now suppose that the fixed point f of an elliptic U is in the upper half-plane, but $f \notin F$, and that the multiplier of U is not a root of unity. For any $z \in F$, the circle fixed by U which passes through z is not completely contained in F. Because the multiplier of U is not a root of unity, the sequence of points z, $U^{-1}z$, $U^{-2}z$,..., is dense on this fixed circle. Thus there exist values of k for which $U^{-k}z \notin F$.

The only possibility remaining is that U is elliptic, has a fixed point in F, and its multiplier is not a root of unity.

The remainder of the proof uses

LEMMA 7. *Let $U = [\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}]$ be an elliptic matrix of determinant P^g, $g > 0$, whose multiplier is not a root of unity.*

(i) *The eigenvalues φ and φ' of U are P-adic integers, one of which is $\equiv 0 \pmod{P}$. The other is $\equiv a + b' \pmod{P}$.*

(ii) $U^k \equiv (a+b')^{k-1} U \pmod{P}$ *for* $k \geqslant 1$.

Proof. See [4, Sect. 9].

Proof of Theorem 6. (continued). Calculations from [4, Sect. 8] show that the multiplier of U is not a root of unity if $a + b' \not\equiv 0 \pmod{P}$. In fact, this stronger condition $a + b' \not\equiv 0 \pmod{P}$ is necessary for a matrix U to be a period matrix of an expansion $\Omega(\zeta)$. For, because $\Omega(\zeta)$ is periodic, there is an index n such that $T_{n+kg}(\zeta) = T_n(\zeta)\, U^k$ for all $k \geqslant 0$. By (ii) of Lemma 7, $T_{n+kg}(\zeta) = T_n(\zeta)\, U^k \equiv (a+b')^{k-1} T_n(\zeta)\, U \pmod{P}$ for all $k \geqslant 1$. If $a + b' \equiv 0 \pmod{P}$ then every entry of $T_{n+kg}(\zeta)$ is divisible by P, which violates the condition of Theorem 1. This completes the proof of the theorem.

7. PURELY PERIODIC EXPANSIONS

The next proposition gives necessary and sufficient conditions for when a matrix U can be the period matrix of a purely periodic expansion $\Omega(\zeta)$.

PROPOSITION 8. *There exists a $\lambda \in F$ and an irrational P-adic integer ζ for which $\Omega(\zeta)$ is purely periodic with period matrix $U = [\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}]$ of determinant P^g, $g > 0$, if and only if U is elliptic with a fixed point in F, $a + b' \not\equiv 0$ $\pmod{P}$ and $(b, b') = 1$.*

Proof. If, for some $\lambda \in F$, $\Omega(\zeta)$ is purely periodic with period matrix $U = [\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}]$, then by Theorem 6, U is elliptic, has a fixed point in F and $a + b' \not\equiv 0 \pmod{P}$. Note that $T_{kg}(\zeta) = U^k$ for all $k \geqslant 0$ and that, by (ii) of Lemma 7,

$$T_{kg}(\zeta) = U^k = (a+b')^{k-1} \begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \pmod{P}, \qquad k \geqslant 1.$$

The lower entries of T_{kg}

$$q_{kg} \equiv (a+b')^{k-1} b \pmod{P} \qquad \text{and} \qquad q'_{kg} \equiv (a+b')^{k-1} b' \pmod{P}$$

must be relatively prime by the conditions of Theorem 1. If $(b, b') > 1$ then, because $ab' - a'b = P^g$, their common factor must be a power of P.

But then the above congruences would yield that both q_{kg} and q'_{kg} are $\equiv 0 \pmod{P}$, which would contradict that $(q_{kg}, q'_{kg}) = 1$. Thus, $(b, b') = 1$.

Conversely, suppose that $U = \left[\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}\right]$ has determinant P^g, $g > 0$, is elliptic with a fixed point $f \in F$, $a + b' \not\equiv 0 \pmod{P}$ and $(b, b') = 1$. Let $\lambda = f$. Then $(b, b') = 1$ and $U^{-1}\lambda = \lambda \in F$, so by the corollary to Theorem 1, $U = T_g(\xi)$ for some ξ. Set

$$\Omega_i = \Omega_i(\xi), \qquad 1 \leqslant i \leqslant g,$$
$$\Omega_{kg+i} = \Omega_i, \qquad 1 \leqslant i \leqslant g \text{ all } k \geqslant 0.$$

Note that $\Omega_{kg+1} \cdots \Omega_{(k+1)g} = \Omega_1 \cdots \Omega_g = U$ for all $k \geqslant 0$. Theorem 1 is used to show that $\{\Omega_n\}_{n=1}^{\infty} = \Omega(\zeta)$ for some ζ. For every $n > 0$, $T_n = \Omega_1 \cdots \Omega_n = \left[\begin{smallmatrix} p_n & p'_n \\ q_n & q'_n \end{smallmatrix}\right]$, it must be verified that $T_n^{-1}\lambda \in F$ and $(q_n, q'_n) = 1$. There exist integers j and k with $0 < j \leqslant g$ and $k \geqslant 0$ satisfying $n = kg + j$. Then

$$T_n = \Omega_1 \cdots \Omega_n = (\Omega_1 \cdots \Omega_g)^k \Omega_1 \cdots \Omega_j = U^k \Omega_1 \cdots \Omega_j.$$

Thus

$$T_n^{-1}\lambda = (\Omega_1 \cdots \Omega_j)^{-1} U^{-k} \lambda = (T_j(\xi))^{-1} \lambda \in F.$$

So suppose that $(q_n, q'_n) > 1$. Then $q_n \equiv q'_n \equiv 0 \pmod{P}$. Let $M = \Omega_{j+1} \cdots \Omega_g = \left[\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right]$. Denote the entries of $U^{k+1} = T_n M = \left[\begin{smallmatrix} a^* & a'^* \\ b^* & b'^* \end{smallmatrix}\right]$. By part (ii) of Lemma 7,

$$U^{k+1} \equiv (a + b')^k \begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \pmod{P},$$

so

$$b^* \equiv (a + b')^k b \pmod{P} \qquad \text{and} \qquad b'^* \equiv (a + b')^k b' \pmod{P}.$$

Since $(b, b') = 1$ and $(a + b') \not\equiv 0 \pmod{P}$, it follows that $(b^*, b'^*) = 1$. But $b^* = q_n x + q'_n y \equiv 0 \pmod{P}$ and $b'^* = q_n x' + q'_n y' \equiv 0 \pmod{P}$, a contradiction. Thus $(q_n, q'_n) = 1$ for all n. This completes the proof that $\{\Omega_n\}$, the purely periodic sequence with period matrix U, is $\Omega(\zeta)$ for some ζ when $\lambda = f$.

The next proposition and its corollary indicate that if $\Omega(\zeta)$ is purely periodic for an irrational ζ, then ζ is the root of a specified quadratic equation.

PROPOSITION 9 (Mahler). *Let $U = \left[\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}\right]$ be an elliptic matrix of determinant P^g for which $a + b' \not\equiv 0 \pmod{P}$. Let the entries of U^k for $k \geqslant 1$ be denoted by $U^k = \left[\begin{smallmatrix} a_k & a'_k \\ b_k & b'_k \end{smallmatrix}\right]$ and let φ denote the eigenvalue of U which is $\equiv 0$*

(mod P). *(φ' denotes the other eigenvalue of U.) Then* $-a_k/b_k$ *converges P-adically to* $(\varphi - a)/b$.

Proof. See [4, Sect. 9].

COROLLARY. *Let ζ be an irrational P-adic integer. There exists a $\lambda \in F$ such that $\Omega(\zeta)$ is purely periodic if and only if ζ is a root of*

$$b^2X^2 + b(a-b')X + (P^g - ab') = 0,$$

where a, b, and b' are integers satisfying $ab' \equiv P^g \pmod b$, $a + b' \equiv 0 \pmod P$, $(b, b') = 1$ *and* $(a - b' + \sqrt{(a-b')^2 - 4P^g})/2b \in F$.

Proof. One of the roots of the above equation is $\zeta = (\varphi - a)/b$, where φ is the eigenvalue that is $\equiv 0 \pmod P$ of the integer matrix $U = \left[\begin{smallmatrix} a & (ab' - P^g)/b \\ b & b' \end{smallmatrix}\right]$. By Proposition 8, the conditions are necessary and sufficient for U to be the period matrix of a purely periodic $\Omega(\zeta)$ for some λ and some ζ; Proposition 9 verifies that this purely periodic $\Omega(\zeta)$ is in fact the expansion for $\zeta = (\varphi - a)/b$.

The other root of the equation has a purely periodic sequence for the same λ with period $P^g U^{-1} = \left[\begin{smallmatrix} b' & (P^g - ab')/b \\ -b & a \end{smallmatrix}\right]$.

8. THE λ-DEPENDENCE OF PERIODICITY

For P-adic integers ζ which have $\Omega(\zeta)$ periodic for some λ, it is possible the conclude the following corollary to Proposition 9.

COROLLARY (Mahler). *Suppose $\Omega(\zeta)$ is periodic with period matrix $U = \left[\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}\right]$. If the determinant of U is P^g then for some $n \geqslant 0$, $T_{n+kg}(\zeta) = TU^k$ for all $k \geqslant 0$, where $T_n(\zeta) = T = \left[\begin{smallmatrix} p & p' \\ q & q' \end{smallmatrix}\right]$. Let φ denote the eigenvalue of U that is* $\equiv 0 \pmod P$. *Then $\zeta = \hat{T}(\xi) = (-p\xi + p')/(q\xi - q')$, where $\xi = (\varphi - a)/b$.*

Proof. Immediate from the proposition and Lemma 3.

Since the eigenvalues of a period matrix U are quadratic irrationals, it follows that if $\Omega(\zeta)$ is periodic, then ζ is a quadratic irrational. However, this periodicity is very λ dependent. For example, consider the simplest case: for some λ and ζ, suppose $\Omega_n(\zeta) = \Omega = \left[\begin{smallmatrix} \alpha & \alpha' \\ \beta & \beta' \end{smallmatrix}\right]$ for all values of n, where Ω is an elliptic matrix of determinant P with a fixed point in F, and whose multiplier is not a root of unity. Then $T_n(\zeta) = \Omega^n$. By Theorem 1, the sequence of points λ, $\Omega^{-1}\lambda$, $\Omega^{-2}\lambda, \ldots$, all lie in F. Because it is elliptic, the points all lie on a circle fixed by the transformation Ω. Because the multiplier of Ω is not a root of unity, the sequence of points is dense on this circle. Consequently, the entire circle must be contained in F.

If a λ' is chosen so that the circle fixed by Ω containing λ' is not completely contained in F, then for infinitely many values of n,

$$\Omega^{-n}\lambda' \notin F. \tag{10}$$

Furthermore, because the multiplier of Ω is not a root of unity, the values of n for which (10) occurs do not form a regular periodic pattern. By Lemma 4, $T_n(\zeta) = \Omega^n S_n$ for some unimodular matrix S_n, and $S_n \neq [\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}]$ if n is a value for which (10) occurs. Hence $\Omega_n(\zeta) = S_{n-1}^{-1}\,\Omega S_n$, and for this λ', the sequence $\Omega(\zeta)$ is not periodic.

9. Essential Periodicity and λ-Relatedness

Notation. Let ζ be a P-adic integer. The sequences $\Omega(\zeta)$, $T(\zeta)$, and $z(\zeta)$ for a particular λ will be denoted by $\Omega_\lambda(\zeta)$, $T_\lambda(\zeta)$, and $z_\lambda(\zeta)$; the elements of these sequences will be denoted $\Omega_{\lambda,n}(\zeta)$, $T_{\lambda,n}(\zeta)$, and $z_{\lambda,n}(\zeta)$.

Definition. A P-adic integer ζ is said to be *essentially periodic* if there is a $\lambda \in F$ for which $\Omega_\lambda(\zeta)$ is periodic; ζ is *essentially purely periodic* if there is a $\lambda \in F$ for which $\Omega_\lambda(\zeta)$ is purely periodic.

From the second corollary to Proposition 9, if an irrational P-adic integer ζ is essentially periodic, then ζ is a quadratic irrational. In this and the following section, a characterization of the quadratic irrational ζ which are essentially periodic is developed. Not all quadratic irrationals have the specified characteristics and are therefore never periodic.

Definition. Let λ be a point in F and let ξ be a P-adic integer. A P-adic integer ζ is *λ-related* to ξ if and only if there exists a $\lambda' \in F$ and non-negative integers m and n such that $\Omega_{\lambda',m+i}(\zeta) = \Omega_{\lambda,n+i}(\xi)$ for all $i > 0$.

Remark. This is not an equivalence relation; however, if ζ is λ-related to ξ using λ', then ξ is λ'-related to ζ.

Proposition 10. *An irrational P-adic integer ζ is essentially periodic if and only if there exists a $\lambda \in F$ and a P-adic integer ξ with $\Omega_\lambda(\xi)$ purely periodic such that ζ is λ-related to ξ.*

Proof. The sufficiency of the condition follows from the definitions of essential periodicity and λ-relatedness. For the converse, let $\lambda' \in F$ be such that $\Omega_{\lambda'}(\zeta)$ is periodic with period matrix $U = [\begin{smallmatrix}a & a'\\ b & b'\end{smallmatrix}]$ of determinant P^g, $g > 0$. For some n and all $k \geqslant 0$, $T_{\lambda',n+kg}(\zeta) = T_{\lambda',n}(\zeta)\,U^k$.

First, note that it is always possible to choose n so that $(b, b') = 1$. For suppose $(b, b') > 1$. Then $P \mid b$ and $P \mid b'$. Denote the entries of $T_{\lambda',n}(\zeta) =$

$\left[\begin{smallmatrix} p_n & p'_n \\ q_n & q'_n \end{smallmatrix}\right]$. Then $|q_n\zeta + p_n|_P \leqslant P^{-n}$ and $|q'_n\zeta + p'_n|_P \leqslant P^{-n}$. Also $T_{\lambda',n+g}(\zeta) = T_{\lambda',n}(\zeta)\, U$ will have

$$\begin{aligned} |(q_n a + q'_n b)\,\zeta + (p_n a + p'_n b)|_P &\leqslant P^{-(n+g)} \quad \text{and} \\ |(q_n a' + q'_n b')\,\zeta + (p_n a' + p'_n b')|_P &\leqslant P^{-(n+g)}. \end{aligned} \tag{11}$$

U is a period matrix so by Theorem 6, $P \nmid (a + b')$. Since it is assumed that $P | b'$, $P \nmid a$.

Note that at least one of $|q_n\zeta + p_n|_P$, $|q'_n\zeta + p'_n|_P$ is $= P^{-n}$ because

$$\begin{aligned} P^{-n} = |p_n q'_n - q_n p'_n|_P &= |q'_n(q_n\zeta + p_n) - q_n(q'_n\zeta + p'_n)|_P \\ &\leqslant \max\{|q_n\zeta + p_n|_P, |q'_n\zeta + p'_n|_P\} \leqslant P^{-n}. \end{aligned}$$

Suppose that $|q_n\zeta + p_n|_P = P^{-n}$. Then

$$|(q_n a + q'_n b)\,\zeta + (p_n a + p'_n b)|_P \leqslant \max\{|a(q_n\zeta + p_n)|_P, |b(q'_n\zeta + p'_n)|_P\};$$

in fact, equality would hold because

$$|a(q_n\zeta + p_n)|_P = P^{-n} \qquad \text{and} \qquad |b(q'_n\zeta + p'_n)|_P \leqslant P^{-n-1}$$

are not equal. Thus

$$|(q_n a + q'_n b)\,\zeta + (p_n a + p'_n b)|_P = |a(q_n\zeta + p_n)|_P = P^{-n}.$$

But this contradicts (11), hence $|q_n\zeta + p_n|_P < P^{-n}$, and $|q'_n\zeta + p'_n|_P = P^{-n}$.

Now, because ζ is not rational, $|q_n\zeta + p_n|_P = P^{-l}$ for some $l > n$. The set of pairs (p, q) which satisfy $|q\zeta + p|_P \leqslant P^{-l}$ is a subset of those which satisfy $|q\zeta + p|_P \leqslant P^{-n}$. Since (p_n, q_n) is the pair satisfying $|q\zeta + p|_P \leqslant P^{-n}$ for which the value of $\Phi(p, q)$ is minimized, and (p_n, q_n) also satisfies $|q\zeta + p|_P \leqslant P^{-l}$, $\Phi(p_n, q_n)$ must be the minimum value of $\Phi(p, q)$ for all pairs (p, q) which satisfy $|q\zeta + p|_P \leqslant P^{-l}$. Thus

$$(p_n, q_n) = (p_l, q_l) \qquad \text{or} \qquad (p'_l, q'_l) \qquad \text{or} \qquad (p_l - p'_l, q_l - q'_l)$$

by Proposition 2. The last case occurs if and only if $z_l = (-1 + \sqrt{-3})/2$. The matrix T_l is unique up to multiplication on the right by $\left[\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right]^{\pm 1}$ by the proof of Proposition 2. So in this case it may be assumed that $(p_n, q_n) = (p_l, q_l)$.

If $(p_n, q_n) = (p'_l, q'_l)$, then

$$\begin{bmatrix} p_n & p'_n \\ q_n & q'_n \end{bmatrix} \Omega_{n+1} \cdots \Omega_l = \begin{bmatrix} p_l & p'_l \\ q_l & q'_l \end{bmatrix} = \begin{bmatrix} p_l & p_n \\ q_l & q_n \end{bmatrix}$$

implies that $\Omega_{n+1} \cdots \Omega_l = [\begin{smallmatrix} c & 1 \\ -P^{l-n} & 0 \end{smallmatrix}]$ for some integer c. But $z_n = (\Omega_{n+1} \cdots \Omega_l)\, z_l$ which is impossible because for $l > n$

$$\begin{bmatrix} c & 1 \\ -P^{l-n} & 0 \end{bmatrix} F \cap F = \varnothing.$$

(The image of F under the action of this transformation is completely contained inside a translation along the real axis of a half circle of radius $P^{n-l} < 1$.) So it is impossible to have $(p_n, q_n) = (p'_l, q'_l)$.

It can be assumed then that $(p_n, q_n) = (p_l, q_l)$. If the period matrix is then written as

$$U^{\#} = \Omega_{\lambda',l+1}(\zeta) \cdots \Omega_{\lambda',l+g}(\zeta) = \begin{bmatrix} a^{\#} & a'^{\#} \\ b^{\#} & b'^{\#} \end{bmatrix}$$

$(b^{\#}, b'^{\#}) = 1$ as desired. For if not, repeating the above procedure would lead to the contradictory statement that $|q_l \zeta + p_l| < P^{-l}$. So without loss of generality it may be assumed that n can be chosen so that the period matrix $U = [\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}]$ has $(b, b') = 1$.

With such an n, set

$$\lambda = z_{\lambda',n}(\zeta) = (T_{\lambda',n}(\zeta))^{-1} \lambda'.$$

Define the sequence $\{\Omega_m\}$ by

$$\Omega_m = \Omega_{\lambda',n+m}(\zeta) \qquad \text{all } m > 0.$$

This sequence is $\Omega_\lambda(\xi)$ for some ξ by Theorem 1. The conditions are verified as follows. For $T_m = \Omega_l \cdots \Omega_m = [\begin{smallmatrix} p_m & p'_m \\ q_m & q'_m \end{smallmatrix}]$, it must be that $(q_m, q'_m) = 1$. For if this failed for some m_0 then $P \mid (q_{m_0}, q'_{m_0})$; in fact, for all $m > m_0$,

$$P \mid q_m \qquad \text{and} \qquad P \mid q'_m. \tag{12}$$

But for values of m which are multiples of g,

$$T_m = T_{kg} = U^k \equiv (a + b')^{k-1} U \pmod{P}$$

by Lemma 7. U is a period matrix for ζ so by Theorem 6, $P \nmid (a + b')$. And since $(b, b') = 1$, (12) is impossible for $m = kg$. Also,

$$\begin{aligned} T_m^{-1} \lambda &= (\Omega_m^{-1} \cdots \Omega_1^{-1})(T_{\lambda',n}(\zeta))^{-1} \lambda' \\ &= (\Omega_{\lambda',n+m}(\zeta))^{-1} \cdots (\Omega_{\lambda',n+1}(\zeta))^{-1} (T_{\lambda',n}(\zeta))^{-1} \lambda' \\ &= (T_{\lambda',n+m}(\zeta))^{-1} \lambda' \in F. \end{aligned}$$

The ξ defined by this sequence is essentially purely periodic with period matrix U for the specified λ. (The value of ξ is given by Proposition 9). And ζ is λ-related to this ξ.

A more thorough study of λ-relatedness will better explain which P-adic integers are essentially periodic.

10. Necessary and Sufficient Conditions for λ-Relatedness

In general, if ζ is λ-related to ξ, for $\lambda' \in F$ and nonnegative integers m and n

$$\Omega_{\lambda',m+i}(\zeta) = \Omega_{\lambda,n+i}(\xi) \qquad \text{all } i > 0.$$

Then, there exists an integer matrix T of determinant P^{m+n-2k}, for some $0 \leqslant k \leqslant \min\{m, n\}$, for which

$$P^{n-k} T_{\lambda',m+i}(\zeta) = T T_{\lambda,n+i}(\xi) \qquad \text{all } i \geqslant 0. \tag{13}$$

This T can be constructed as

$$T = P^{-k}(T_{\lambda',m}(\zeta))(P^n T_{\lambda,n}(\xi))^{-1}),$$

where k is sufficiently large so that the entries of T have no common factor.

Conversely, if there is a $\lambda' \in F$ and nonnegative integers m, n, and k, with $k \leqslant \min\{m, n\}$, such that an integer matrix T of determinant P^{m+n-2k} satisfies (13), then ζ is λ-related to ξ.

The following theorem gives necessary and sufficient conditions which a matrix T must satisfy in order to λ-relate a P-adic integer ζ to a P-adic integer ξ. The proof is similar to that of Theorem 5.

Theorem 11. *Let ζ and ξ be P-adic integers, let $\lambda \in F$, and let m, n, and k be nonnegative integers with $k \leqslant \min\{m, n\}$. Let T be an integer matrix of determinant P^{m+n-2k}. Then ζ is λ-related to ξ if and only if, for $T = \left[\begin{smallmatrix} p & p' \\ q & q' \end{smallmatrix}\right]$,*

(i) $\zeta = \hat{T}\xi$,

(ii) $T^{-1}F \cap (\bigcap_{i=0}^{\infty} T_{\lambda,n+i}(\xi)\, F) \neq \varnothing$,

(iii) $\min\{\operatorname{ord}_P(q\xi - q'), \operatorname{ord}_P(p\xi - p')\} = \operatorname{ord}_P(q\xi - q') = n - k$.

Proof. Let T be as specified and assume that ζ is λ-related to ξ. Then there is a $\lambda' \in F$ for which (13) is satisfied. The proof that this implies conditions (i) and (iii) is identical to the proof for the identical conditions appearing in Theorem 5. The condition (ii) is demonstrated as follows. For all $j \geqslant 0$, $\lambda' \in T_{\lambda',j}(\zeta)\, F$; thus for all $i \geqslant 0$

$$\lambda' \in T_{\lambda',m+i}(\zeta)\, F = P^{n-k} T_{\lambda',m+i}(\zeta)\, F = T T_{\lambda,n+i}(\xi)\, F$$

so that $T^{-1}\lambda' \in T_{\lambda,n+i}(\xi)\,F$. Also $\lambda' \in F$ so $T^{-1}\lambda' \in T^{-1}F$. Thus the set

$$T^{-1}F \cap \left(\bigcap_{i=0}^{\infty} T_{\lambda,n+i}(\xi)\,F\right) \neq \varnothing$$

because it contains $T^{-1}\lambda'$.

Now assume that conditions (i)–(iii) are satisfied. The conditions (i) and (iii) are used exactly as in Theorem 5, with elements of $T_\lambda(\xi)$ replacing those of $T(\xi)$, to verify that, for any $j \geqslant n-k$,

$$T^*_j = \frac{1}{P^{n-k}}\,TT_{\lambda,j}(\xi) = \begin{bmatrix} p^*_j & p'^*_j \\ q^*_j & q'^*_j \end{bmatrix}$$

is an integer matrix of determinant P^{m+j-n} and that

$$|q^*_j\zeta + p^*_j|_P \leqslant P^{n-m-j} \qquad \text{and} \qquad |q'^*_j\zeta + p'^*_j|_P \leqslant P^{n-m-j}.$$

Let μ be any point in $T^{-1}F \cap (\bigcap_{i=0}^{\infty} T_{\lambda,n+i}(\xi)\,F)$, which is non-empty by (ii); let $\lambda' = T\mu$. Then by Lemma 4, $T^*_j = T_{\lambda',m+j-n}(\zeta)\,S_{j-n}$, for some unimodular matrix S_{j-n}.

Set $j = n+i$. Then

$$TT_{\lambda,n+i}(\xi) = P^{n-k}T^*_{n+i} = P^{n-k}T_{\lambda',m+i}(\zeta)\,S_i.$$

The choice of λ' guarantees that, for all $i \geqslant 0$,

$$\lambda' \in TT_{\lambda,n+i}(\xi)\,F = T_{\lambda',m+i}(\zeta)\,S_iF$$

so that

$$(T_{\lambda',m+i}(\zeta))^{-1}\lambda' \in S_iF.$$

But by Theorem 1,

$$(T_{\lambda',m+i}(\zeta))^{-1}\lambda' \in F.$$

Thus

$$z_{\lambda',m+i}(\zeta) = (T_{\lambda',m+i}(\zeta))^{-1}\lambda' \in S_iF \cap F$$

for all $i \geqslant 0$. The set $S_iF \cap F$ is nonempty if $S_i = \pm\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, or $S_i = \pm\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$ when $S_iF \cap F = \sqrt{-1}$, or $S_i = \pm\left[\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right]^{\pm 1}$ when $S_iF \cap F = (-1+\sqrt{-3})/2$. This situation occurred also at the end of the proof of Theorem 5. The arguments using Proposition 2 that appeared there now allow the conclusion

$$TT_{\lambda,n+i}(\xi) = P^{n-k}T_{\lambda',m+i}(\zeta)\,S_i = P^{n-k}T_{\lambda',m+i}(\zeta).$$

This completes the proof.

11. Conclusions

P-adic integers ζ and ξ are equivalent when their expansions $\Omega(\zeta)$ and $\Omega(\xi)$ eventually agree for a fixed λ (associated to a form Φ). A characterization of this natural equivalence relation between P-adic integers is given in Theorem 5; the P-adic integers are related by a certain class of linear fractional transformations.

This equivalence relation fails to characterize which P-adic integers have periodic expansions, because a P-adic integer may have a periodic expansion for some λ but not for others. A generalization of the notion of equivalence to λ-relatedness allows a characterization of P-adic integers which can have periodic expansions, i.e., are essentially periodic. The characterization is given by Theorem 11, Proposition 10, and the first corollary to Proposition 9.

In particular, the characterization can be used to show that *not* all quadratic irrational P-adic integers are essentially periodic. In Proposition 10, it is shown that an irrational P-adic integer ζ is essentially periodic if and only if it is λ-related to some irrational P-adic integer ξ for which $\Omega_\lambda(\xi)$ is purely periodic. The corollary to Proposition 9 shows that ξ is essentially purely periodic when ξ is a solution of a particular shape of quadratic equation which cannot have positive discriminant; hence if $\Omega_\lambda(\xi)$ is purely periodic, ξ cannot have positive discriminant. By Theorem 11, if ζ is λ-related to ξ then $\zeta = T\xi$ for an integer matrix T; it follows that ζ cannot have positive discriminant. Thus, no quadratic irrational P-adic integer with positive discriminant can be essentially periodic.

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